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Hydromagnetic stability of a self-gravitating composite plasma in a two dimensional horizontal magnetic field

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Abstract : The hydromagnetic stability of self-gravitating incompressible viscous finitely conducting partially ionized plasma permeated by a uniform two dimensional horizontal magnetic field has been investigated here. A variational principle is shown to characterize the problem. Proper solution has been obtained by making use of the existence of the variational principle for a semi infinite plasma in which density has one dimensional stratification along the vertical. The dispersion relation has been derived and solved numerically. It is found that the viscosity and collision with neutrals have stabilizing influence while finite conductivity has destabilizing influence on the growth rate of unstable mode of disturbance.

Keywords : Composite plasma, hydromagnetic stability, viscosity, variational principle.

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1. Introduction

The hydromagnetic stability of a magnetized plasma of varying density is of considerable importance in several astrophysical and geophysical situations. This stability problem has been investigated by several researchers [1,2] under different physical assumptions and a comprehensive account of these investigations has been given by Chandrasekhar [3]. For more realistic physical situations in astrophysics and geophysics the case of viscous fluids should also be considered. Sundaram [4] has investigated the dynamic stability of a self-gravitating, viscous finitely conducting plasma. Bhatia [5] has included the effect of Hall currents. More recently Gupta and Bhatia [6] have investigated the combined influence of Hall currents and coriolis forces on the instability of a self-gravitating plasma. All these investigations have been carried out for fully-ionized plasma.

Plasmas are not frequently ionized but instead, may be permeated with neutrals and are thus partially ionized so that the interaction between the neutral and the ionized gas becomes important. In cosmic physics there are several situations such as chromospheres, solar photosphere and in cool interstellar clouds where the plasma are partially ionized. Alfver [7] pointed out the importance of such collisions between ionized fluid and neutral gas on the ionization rate in these regions. Lehnert [8] has pointed out the stabilizing influence of frictional effects with neutrals on the instability of a plasma interacting with neutral gas. More recently, Ali and Bhatia [9] have investigated the influence of Hall currents on the instability of a self-gravitating partially ionized plasma.

In all these investigations the prevalent magnetic field is assumed to act either along horizontal or vertical direction. Gupta and Bhatia [10] studied the instability of partially ionized superposed plasmas in a uniform two dimensional horizontal magnetic field. It would therefore be of importance to examine the effects of viscosity, finite conductivity on the hydromagnetic stability of a self-gravitating partially ionized plasma. It is assumed that the plasma is permeated by uniform two dimensional horizontal magnetic field. This aspect forms the basis of this paper wherein the density of the plasma is exponentially stratified along the vertical.

2. Perturbation equations

We consider the horizontal strata of a self-gravitating viscous finitely conducting partially ionized plasma which is in equilibrium under the action of a two dimensional horizontal magnetic field $\boldsymbol{H} = (H_x, H_y, 0)$. It is assumed that the two components of the partially ionized plasma (the ionized fluid and neutral gas) behave as continuums and their steady state velocities are equal. Furthermore we assume that the magnetic field interacts only with the ionized component of the plasma and the frictional force of the neutral gas on the ionized fluid is of the same order as the pressure gradient of the ionized fluid. The force due to gravitational potential, pressure gradient of the neutrals is much less than the frictional force of the ionized fluid. Thus we are considering here only the mutual frictional effects between the neutral gas and the ionized fluid in the present investigation of the stability of self-gravitating partially ionized plasma of varying density.

Under the foregoing assumptions the linearized perturbation equations governing the two components of the partially ionized plasma are

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\delta p + (\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + \delta\rho\nabla\phi + \rho\nabla\delta\phi + \mu\nabla^2\boldsymbol{u} + 2(\nabla\mu.\nabla)\boldsymbol{u} + \rho_d \nu_c(\boldsymbol{u}_d - \boldsymbol{u}), \tag{1}$$

$$\frac{\partial}{\partial t}\boldsymbol{u}_d = \nu_c(\boldsymbol{u}_d - \boldsymbol{u}), \tag{2}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{H}) + \eta\,\nabla^2\boldsymbol{h}, \tag{3}$$

$$\frac{\partial}{\partial t}\delta\rho + (\boldsymbol{u}.\nabla)\rho = 0, \tag{4}$$

$$\nabla^2\delta\phi = -G\delta\rho, \tag{5}$$

$$\nabla.\boldsymbol{u} = 0,\ \nabla.\boldsymbol{h} = 0, \tag{6}$$

where $\boldsymbol{u}\,(u,v,w)$, $\boldsymbol{h}\,(h_x, h_y, h_z)$, $\delta\rho, \delta p$ and $\delta\phi$ are respectively the perturbations in velocity, magnetic field $\boldsymbol{H}$, density ρ, pressure p, gravitational potential ϕ, of the ionized fluid, while ρ_d and $\boldsymbol{u}_d$ denote corresponding quantities for the neutral gas. In the above equations μ is the coefficient of viscosity, G is the gravitational constant while η is magnetic resistivity. The collision frequency between the two components of the plasma is denoted by υ_c.

Analysing the disturbance into normal modes we assume that the perturbed quantitites have the dependence on the space coordinate (x, y, z) and time t of the form

$$F(z) \exp(ik_x x + ik_y y + nt), \tag{7}$$

where $F(z)$ is some function of z, k_x and k_y are the horizontal and vertical wave numbers $(k^2 = k_x^2 + k_y^2)$ and n denotes the rate at which a system departs from equilibrium.

Using eq. (7) in eqs. (1)–(6) and eliminating some of the variables from these equations we finally obtain the equation in w, hz and $\delta\phi$ as

$$n'[\rho\, wk^2 - D(\rho Dw)] + (ik_x H_x + ik_y H_y)(D^2 - k^2) Dh_z + \frac{k^2 w (D\rho)(D\phi)}{n}$$

$$+ k^2 \delta\phi D\rho + \mu (D^2 - k^2)^2 w + D^2 \mu (D^2 + k^2) w + 2(D\mu)(D^2 - k^2) Dw = 0, \tag{8}$$

$$\left[n - \eta (D^2 - k^2)\right] h_z = (ik_x H_x + ik_y H_y) w, \tag{9}$$

$$(D^2 - k^2)\delta\phi = \frac{GD\rho w}{n}, \tag{10}$$

where we have used

$$n' = n\left(1 + \frac{\alpha \upsilon_c}{n + \upsilon_c}\right), \alpha = \frac{\rho_d}{\rho}, D = \frac{d}{dz}. \tag{11}$$

3. Boundary conditions

We have thus a set of three equations in three variables w, h_z and $\delta\phi$ which are to be solved subject to the appropriate boundary conditions.

We assume that the partially ionized plasma is confined between two rigid planes at $z = 0$ and $z = d$ which are both assumed to be ideal conductors. The normal component of velocity must vanish at these boundaries. We must therefore, have

$$\omega = 0 \text{ at } z = 0 \text{ and } z = d. \tag{12a}$$

The electromagnetic conditions at the boundaries require the vanishing of the normal component of the perturbation in the magnetic field i.e.

$$h_z = 0, D^2 w = 0 \text{ at } z = 0 \text{ and } z = d. \tag{12b}$$

Finally, the perturbation in the gravitational potential ϕ must satisfy the condition at the boundaries

$$(D - k)\,\delta\phi = 0, \tag{12c}$$

$$(D + k)\, \delta\phi = 0. \tag{12d}$$

These boundary conditions are obtained by matching the solutions on the boundaries $z = 0$ and $z = d$ in view of the fact that ϕ and the norimal component of grad ϕ are continuous across the surface. For mathematical simplicity we have assumed that the bounding surfaces are rigid and ideal conductors.

4. Variational principle

Suppose that the solutions belonging to characteristic value n_i are w_i, h_i and $\delta\phi_i$ and the solutions corresponding to the value n_j are w_j, h_j and $\delta\phi_j$ where we have dropped the suffix z on h for simplicity.

Multiplying eq. (8) for i by w_j and integrating across the vertical extent L of the plasma we obtain,

$$n'_i \int_L \left[k^2 \rho w_i - D(\rho D w_i)\right] w_j dz + (ik_x H_x + ik_y H_y) \int_L \left[(D^2 - k^2) D h_i\right] w_j dz$$
$$+ \frac{k^2}{n_i} \int_L (D\phi_i)(D\rho)\, w_i w_j dz + k^2 \int_L \left[(D\rho)\delta\phi_i\right] w_j dz + \int_L \mu\, [D^2 - k^2]^2 w_i w_j dz$$
$$+ w_j dz + \int_L D^2 \mu \left[(D^2 + k^2) w_i\right] w_j dz + 2 \int_L (D\mu) \left[(D^2 - k^2) D w_i\right] w_j dz = 0. \tag{13}$$

Performing integration by parts repeatedly and using boundary conditions, on writing $i = j$ we obtain,

$$n' I_1 - n I_2 - \eta I_3 + \frac{k^2}{n} I_4 - \frac{nk^2}{G} \quad I_5 + I_6 = 0 \tag{14}$$

where
$$I_1 = \int_L \rho\ [(Dw)^2 + (k^2 w^2)]\, dz, \tag{15}$$

$$I_2 = \int_L [(Dh)^2 + (k^2 h^2)]\, dz, \tag{16}$$

$$I_3 = \int_L [(D^2 - k^2) h]^2\, dz, \tag{17}$$

$$I_4 = \int_L (D\rho)(D\phi) w^2 dz, \tag{18}$$

$$I_5 = \int [(D\delta\phi)^2 + k^2 (\delta\phi)^2]\, dz, \text{ and} \tag{19}$$

$$I_6 = \int_L \left\{\mu\ [(D^2 + k^2) w]^2 + 4k^2 (Dw)^2\right\} dz. \tag{20}$$

Considering the change δn in n due to arbitrary variations δw and δh through the perturbation form of (9) in w and h respectively, compatible with boundary conditions and proceeding along the usual lines, we can show that, to first order $\delta n = 0$

5. Stratified layer of gravitating plasma

Now we make use of existence of the variational principle to obtain the solution of the problem of a continuously stratified plasma of finite depth d in which the undisturbed density of the conducting component of the partially ionized plasma is given by

$$\rho(z) = \rho_1 \exp(\beta z) \tag{21}$$

where ρ_1 and β are constants. The density ρ_d of the neutral gas is also assumed to be exponentially stratified along the vertical. Furthermore, we assume that the viscosity μ is also vertically stratified exponentially i.e.

$$\mu(z) = v_o \rho_1 \exp(\beta z) \tag{22}$$

where $v_0 = \frac{\mu_0}{\rho_1}$ is constant. The gravitational potential ϕ must satisfy the Poisson's equation. For the density distribution given by eq. (21) the solution of the Poisson's equation

$$\nabla^2 \phi = -G\rho \tag{23}$$

leads to the following distribution for ϕ

$$\phi(z) = \frac{G\rho_1}{\beta^2}\left(e^{-\beta 2} + \beta z + 1\right), \tag{24}$$

appropriate to the boundary conditions. Let us take the trial solution for $h(z)$ as

$$h(z) = X \sin lz \tag{25}$$

where X is a constant and $l = s\pi/d$, s being an integer. If we solve for $w(z)$ from eq. (9), with the assumed value for $h(z)$, we obtain

$$w(z) = \frac{X}{(k.H)}\left[n + \eta\,(l^2 + k^2)\right] \sin lz. \tag{26}$$

Finally, if we solve eq. (10) we obtain

$$(D-k)\,\delta\phi = \frac{G\rho_l\,\beta X e^{\beta z}}{n.\,(k.H)}\left[n + \eta\,(l^2 + k^2)\right] \sin lz. \tag{27}$$

It may be noted that the trial solution for $\delta\phi$ satisfy only one of the boundary conditions in the present problem, as in the case of earlier investigation of Sundaram [4] in the absence of Hall currents. Substituting eqs. (25) – (27) in eq. (14), and evaluating the various integrals contained therein, we obtain the dimensionless form of the dispersion relation. On writing

$$\sigma = \frac{n}{l},\quad A = \eta\, l,\quad B = \upsilon_o l,\quad C = \frac{\upsilon}{l},\quad x = \frac{k}{l},\quad a = \frac{\beta}{l}\quad N = \frac{G\rho_1}{l^2},$$

$$V_1 = \frac{V_x}{1}, \quad V_2 = \frac{V_y}{1}, \quad V_x = \frac{H_x}{\sqrt{\rho_1}} \quad V_y = \frac{H_y}{\sqrt{\rho_1}}, \tag{28}$$

We get

$$P_2 Q_1 \sigma^5 + \left[\left\{(1+\alpha)C + 2A(1+x^2) P_3 Q_1 + B P_4 Q_1\right\}\right]\sigma^4$$
$$+ \left[\{2AC\,(1+\alpha)(1+x^2) + B^2(1+x^2)^2\} P_3 Q_1 + V^2 x^2 (1+x^2) + N P_1 Q_1\right.$$
$$- N P_5 Q_2 + B\,\{2A\,(1+x^2) + C\} P_4 Q_1 \Big]\sigma^3 + \left[A^2 C\,(1+x^2)^2 (1+\alpha) P_2 Q_1\right.$$
$$+ AB\,\{A\,(1+x^2) + C\}(1+x^2) P_4 Q_1 + N\,\{2A\,(1+x^2) + C\} P_1 Q_1$$
$$\left. - N\,\{2A\,(1+x^2) + C\} P_5 Q_2 + V^2 x^2 (1+x^2)\,\{A\,(1+x^2) + C\}\right]\sigma^2$$
$$+ \left[ACx^2(1+x^2)V^2 + NA\,(1+x^2)\,\{A\,(1+x^2) + C\}\,\left\{P_1 Q_1 - P_5 Q_2\right\}\right.$$
$$\left. + A^2 CB\,(1+x^2)^2 P_4 Q_1\right]\sigma + A^2 CN\,(1+x^2)\left\{P_1 Q_1 - P_5 Q_2\right\} = 0 \tag{29}$$

where we have used

$$P_1 = \frac{x^2}{1+a^2/4}, \qquad P_2 = \frac{1+x^2+a^2/2}{1+a^2/2}, \qquad P_3 = \frac{1+x^2+a^2/4}{1+a^2/4},$$
$$P_4 = \frac{2a^2x^2 + (1+x^2)^2}{1+a^2/4}, P_5 = \frac{x^2+a^2}{1+a^2} + \frac{4a^3x}{[1+(a-x)^2]^2}$$
$$Q_1 = \frac{e^{s\pi a}-1}{s\pi a}, \qquad Q_2 = \frac{e^{2s\pi a}-1}{2s\pi a}, \qquad V_2 = (V_1\cos\theta + V_2\sin\theta)^2. \tag{30}$$

Special cases :

(i) $A = 0$

In the case of infinitely conducting plasma the dispersion relation (29) reduces to

$$P_2 Q_1 \sigma^3 + \left[C\,(1+\alpha) + BP_4\right] Q_1 \sigma^2 + \left[\left\{B^2 (1+x^2) P_4 + NP_1\right\} Q_1\right.$$
$$\left. + V^2 x^2\,(1+x^2) - NP_5 Q_2 + BCP_4 Q_1\right]\sigma + NC\,\left\{P_1 Q_1 - P_5 Q_2\right\}$$
$$+ V^2 C x^2\,(1+x^2) = 0. \tag{31}$$

(ii) $A = 0, B = 0$

When the effects of ion viscosity are ignored for infinitely conducting plasma the dispersion relation reduces to

$$P_2 Q_1 \sigma^3 + C\,(1+\alpha)\,Q_1 \sigma^2 + \left[N\,\left\{P_1 Q_1 - P_2 Q_2\right\} + V^2 x^2\,(1+x^2)\right]\sigma$$
$$+ NC\,\left\{P_1 Q_1 - P_5 Q_2\right\} + CV^2 x^2\,(1+x^2) = 0. \tag{32}$$

6. Discussion

The dispersion relation (29) is quite complex. In order to study the various physical effects on the growth rate of unstable mode; we have performed numerical calculation of dispersion relation (29) for several values of parameters which corresponds to the conditions in galaxies

$$\rho_0 = 1.7 \times 10^{-21} \text{ kg.m}^{3}$$

$$G = 6.658 \times 10^{-11} (\text{kg})^{-1} \text{m}^3 \text{s}^{2}$$

The values of the critical wave length for the gravitational instability of a homogeneous infinitely extending plasma are of the order 10^{20} metres. We have, therefore, calculated the roots of the dispersion relation (29) for different values of parameters, A, B and C

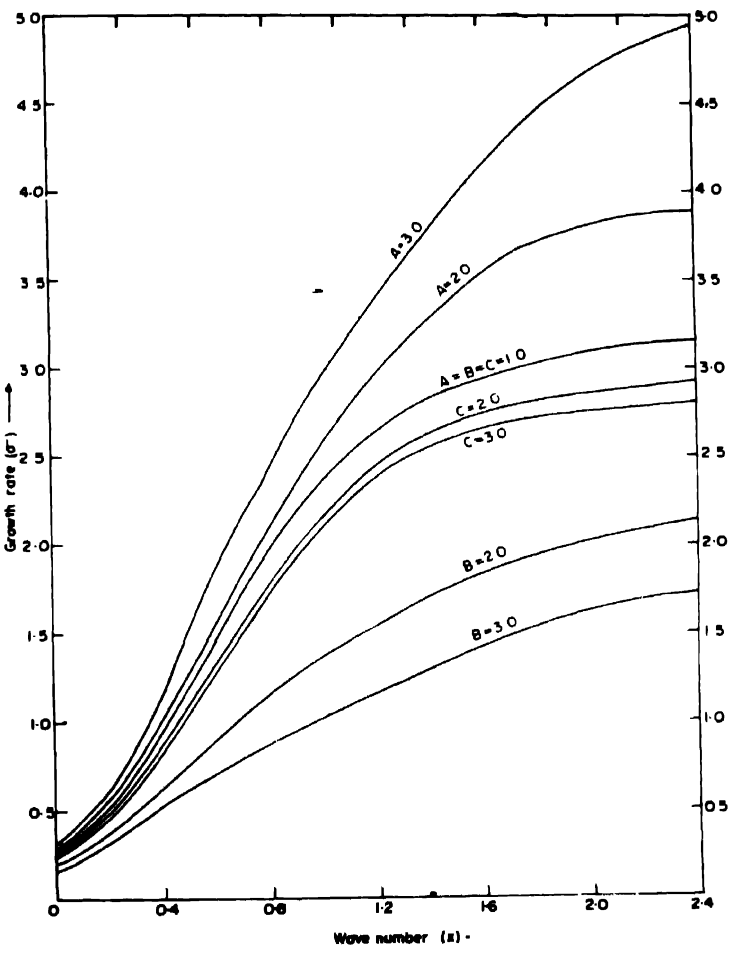

Figure 1. Plot of growth rate (positive real part of σ) against wave number x for different values of A, B and C

characterizing finite conductivity, viscosity and neutral gas friction respectively taking multiples of 10^{-20} as the values for the wave number. The numerical calculations are presented in Figures 1 where have plotted the growth rate (positive real parts after multiplying by 10^4) against wave number x (after multiplying by 10^{20}) for A (after multiplying by 10^4) = 1, 2 and 3 B (after multiplying by 10^4) = 1, 2 and 3 and C (after multiplying by 10^4) = 1, 2 and 3. In all these calculations we have taken a =0.1, s = 1.0, $N = 1$, $\alpha = 0.1$, $V_1 = V_2 = 0.5$, and $\theta = 45°$.

It is clearly seen from Figure 1 that growth rate σ increases as A (finite conductivity) increases for the same x. showing thereby destabilizing influence of finite conductivity. The Figure 1 also shows that as B (ion viscosity) increases, σ (growth rate) decreases for the same x showing thereby stabilizing character of ion viscosity. It is further seen from Figure 1 that growth rate σ decreases, as C (neutral gas friction) increases for the same x, showing stabilizing influence of friction with neutrals. These results are in agreement with earlier observation of Gupta and Bhatia [11] and many others.

We may thus conclude that ion viscosity and neutral gas friction have stabilizing influence while finite conductivity has destabilizing influence on the hydromagnetic stability of a self-gravitating partially ionized plasma in two dimensional horizontal magnetic field.

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